Computers & Structures, Vol. 4, pp. 1313–1321. Pergamon Press 1974. Printed in Great Britain

APPLICATION OF A REFINED PLATE BENDING ELEMENT TO BUCKLING PROBLEMS

B. TABARROK,† R. G. FENTON† and A. M. ELSAIE‡

University of Toronto, Ontario, Canada

Abstract—The buckling stiffness matrix of a refined plate bending element is derived for various continuous and non-uniform distributions of in-plane forces. Elements of this matrix are expressed in an explicit form and can, therefore, be readily used in a finite element computer program for solving plate buckling problems with various loading and edge conditions. A number of problems are solved and the results obtained are compared with analytical and/or numerical results obtained by using other procedures. It is shown that the refined plate bending element is superior to simpler elements when used for solving plate buckling problems.

1. INTRODUCTION

FOR THE analysis of plate bending problems, a large variety of finite element models have been developed. For the classical plate theory the complete satisfaction of the displacement continuity requires the continuity of displacements and the normal slopes along inter-element boundaries. The construction of a displacement shape function for the interior of an element, which will also ensure compatibility along the elemental boundaries, is difficult, even for elements of simple geometry.

For triangular elements it has been found that compatibility requirements can be fulfilled completely if a quintic polynominal is used as a shape function. Such a shape function possesses 21 constants and it can be described by the displacement, and the first two derivatives of the displacement at each vertex of the triangle and 3 normal slopes at midsides of the element. Alternatively by imposition of 3 constraints the midside nodes can be eliminated and an element of 18 degrees-of-freedom is then developed. The term refined element refers to the large number of degrees-of-freedom per element. The elimination of 3 degrees-of-freedom results in a slight loss in accuracy, and further the element becomes direction dependent.

The development of this refined triangular bending element and its variants was undertaken by several investigators independently: by Withum [1], Argyris *et al.* [2, 3], Bell [4], Visser [5], Cowper *et al.* [6–8] and Butlin *et al.* [9]. The results obtained when using the refined elements converge very rapidly, and the accuracy of these results is quite superior to results obtained by using larger number of simpler elements having the same number of degrees-of-freedom.

The refined element has been used for static stress and vibration analysis of plates [6–8]. To complete its application to plate bending problems, we extend its use to plate buckling problems, and present several examples clearly indicating the superiority of the refined element in buckling problems as well.

---

† Associate Professor, Department of Mechanical Engineering.

‡ Post Doctoral Fellow, Department of Mechanical Engineering.

2. THE PLATE BUCKLING PROBLEM

The equations of motion of a plate subjected to in-plane forces, and freely vibrating with frequency, ω, are expressed by the variational statement:

$$\delta(U - W - \omega^2 T) = 0 \tag{1}$$

where U is the strain energy of the plate, $\omega^2 T$ is its complementary kinetic energy, and W is the potential energy of the in-plane forces. All energy terms are functions of the plate displacement, w, and in equation (1) only w is subject to variation.

For finite element formulation

$$U = \Sigma U_e \qquad W = \Sigma W_e \qquad T = \Sigma T_e \tag{2}$$

where U_e, W_e and T_e are energy quantities of the individual elements.

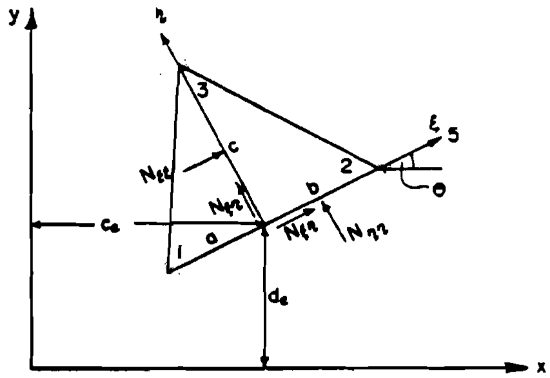


FIG. 1. Coordinate system for one element.

In terms of the local coordinate system of an element, see Fig. 1, the energy functions are:

$$U_e = \iint_e \frac{D}{2}(w^2{}_{,\xi\xi} + w^2{}_{,\eta\eta} + 2\nu w_{,\xi\xi} w_{,\eta\eta} + 2(1-\nu) w^2{}_{,\xi\eta})\, d\xi d\eta \tag{3}$$

$$W_e = \iint_e \tfrac{1}{2}(\bar{N}_{\xi\xi} w^2{}_{,\xi} + \bar{N}_{\eta\eta} w^2{}_{,\eta} + 2\bar{N}_{\xi\eta} w_{,\xi} w_{,\eta})\, d\xi d\eta \tag{4}$$

$$T_e = \iint_e \tfrac{1}{2}\rho t w^2\, d\xi d\eta \tag{5}$$

where $D = Et^3/12(1-\nu^2)$ is the flexural rigidity of the plate, $\bar{N}_{\xi\xi}$, $\bar{N}_{\eta\eta}$ and $\bar{N}_{\xi\eta}$ are prescribed in-plane forces per unit length, ρ is the mass per unit area of the plate material, and the integrals are taken over the total surface area of the element. It is now assumed that the normal displacement of the plate is given by a quintic polynomial:

$$w(\xi, \eta) = a_1 + a_2\xi + a_3\eta + a_4\xi^2 \ldots + a_{21}\eta^5 \tag{6}$$

or

$$w = \sum_{i=1}^{21} a_i \xi^{m_i} \eta^{n_i} \tag{7}$$

where the indices m_i and n_i can be identified from equation (6), and a_i are undetermined coefficients. In [7] the coefficient of the term $\xi^4\eta$ in equation (6) is equated to zero ensuring that the slope normal to the edge $\eta=0$ is a cubic polynomial, in addition two further constraints are imposed to ensure that the slopes normal to the remaining two edges of the element also take the form of cubic polynomials. Thus the matrix which relates the 20 unspecified coefficients a_i to the 18 nodal quantities is rectangular, and it is denoted by $[T_2]$.

In terms of the global coordinate system the strain energy of an element is:

$$U_e=\tfrac{1}{2}D\{w\}^t[R]^t[T_2]^t[K][T_2][R][w] \tag{8}$$

where $[K]$ is the stiffness matrix of the element and $[R]$ is the rotation matrix that transforms the local coordinates to global coordinates. Matrices $[K]$, $[T_2]$ and $[R]$ are given in [7]. The kinetic energy of the element in terms of the global coordinate system is:

$$T_e=\tfrac{1}{2}\omega^2\rho t\{w\}^t[R]^t[T_2]^t[M][T_2][R]\{w\} \tag{9}$$

where $[M]$ is the consistent mass matrix. This matrix is also given in [7].

The in-plane forces $\bar{N}_{\xi\xi}$, $\bar{N}_{\eta\eta}$ and $\bar{N}_{\xi\eta}$ are usually prescribed in the global coordinate system and in the present analysis we assume that they may take the following forms:

$$\begin{aligned} N_{xx}&=N(\alpha_1+\alpha_2x+\alpha_3y+\alpha_4xy)\\ N_{yy}&=N(\beta_1+\beta_2x+\beta_3y+\beta_4xy)\\ N_{xy}&=N(\gamma_1-\beta_3x-\alpha_2y-.5\beta_2x^2-.5\alpha_4y^2) \end{aligned} \tag{10}$$

where N is the critical load at which buckling takes place. It is obvious that the prescribed in-plane forces causing the plate to buckle need not be uniformly distributed over the edges of the plate. Equations (10) implicitly satisfy the inplane equilibrium of the plate and by specifying the constants $\alpha_1 \ldots \gamma_1$ a variety of force distributions can be described for which solutions may be obtained. Components of these forces in the local coordinate system are obtained as:

$$\begin{aligned} N_{\xi\xi}&=N(A_1+A_2x+A_3y+A_4xy+A_5x^2+A_6y^2)\\ N_{\eta\eta}&=N(B_1+B_2x+B_3y+B_4x+yB_5x^2+B_6y^2)\\ N_{\xi\eta}&=N(C_1+C_2x+C_3y+C_4xy+C_5x^2+C_6y^2). \end{aligned} \tag{11}$$

For the values of A_1 to C_6 in terms of α_1 to γ_1 see the appendix.

The transformation from the global to the local coordinate system can be completed using the following equations:

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{Bmatrix} \xi \\ \eta \end{Bmatrix} + \begin{Bmatrix} c_e \\ d_e \end{Bmatrix} \tag{12}$$

where c_e and d_e are the cartesian coordinates of the origin of the local coordinate system given in global coordinates, see Fig. 1.

Expressing equation (11) in terms of ξ and η and also using equation (7) in the expression for W_e in equation (4) and carrying out the integration over the area of the element one can obtain the following discretised expression for W_e in terms of a_i

$$W_e=\tfrac{1}{2}\{a\}^t[K_b]\{a\}. \tag{13}$$

In terms of the global coordinate system W_e is

$$W_e=\tfrac{1}{2}\{w\}^t[R]^t[T_2]^t[K_b][T_2][R]\{w\} \tag{14}$$

where elements of $[K_b]$ are given in the appendix.

The expressions for U_e, W_e and T_e can now be computed in terms of the element parameters, and the usual procedure of assembly of elemental matrices is used to obtain the global equations of the complete plate. These equations can be written in the following eigenvalue form

$$([K]-[K_b]-\omega^2[M])\{q\}=\{0\} \tag{15}$$

where $\{q\}$ is the vector of the global variables.

3. NUMERICAL EXAMPLES

A number of buckling problems of square plates have been solved using the elemental matrices outlined in this paper and a summary of the results obtained is given in Table 1. Eigenvalues and their associated vectors were computed from equation (15) for three different cases:

(i) free natural vibration—(zero in-plane forces)—$\lambda_v=\rho t\omega^2L^4/D$
(ii) pure buckling—(no vibration)—$\lambda_b=NL^2/\pi^2D$
(iii) combined buckling and vibration—(in-plane forces equal in magnitude to half the buckling loads)—$\lambda_{vb}=\rho t\omega^2L^4/D$ where L is the side length of the square plate.

Results were computed for square plates having a length-to-thickness ratio of 0·01, and for different boundary conditions. Two arrangements of the elements were considered—the Q arrangement as used in [7], and a new arrangement which will be referred to as the PQ arrangement, see Fig. 2.

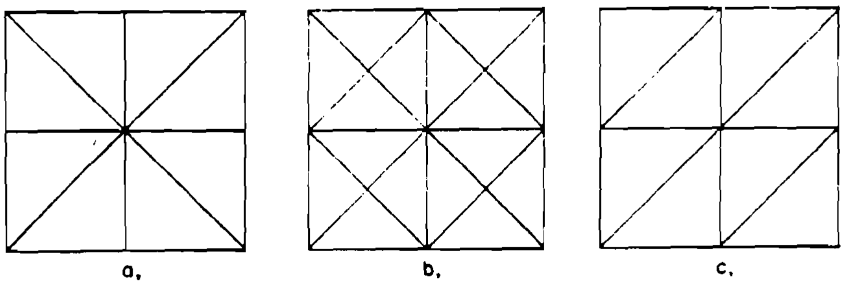
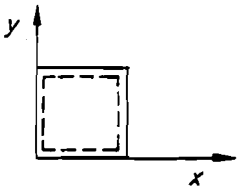
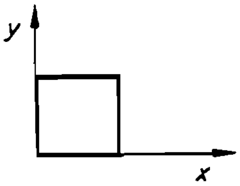
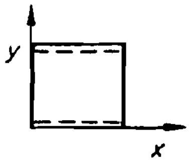
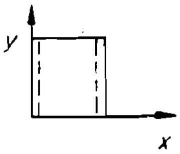
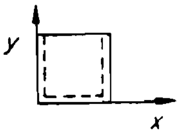
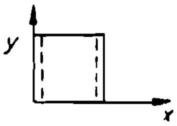


FIG. 2. Element arrangements: (a) Q arrangement; (b) PQ arrangement; (c) nonsymmetric arrangement.

TABLE 1. Buckling and vibration eigenvalues of a square plate under various in-plane loading configurations

Example number	Arrangement	Configuration	λ_v	λ_b	λ_{vb}	Loading
1	Q dof = 22		389·83 Exact = 389·63 [6–8]	4·0019 Exact = 4·00 [17]	194·92	Constant N
2				±9·5998 Exact = 9·34 [17]	306·97	Constant N_{xy}
3				7·8221 Exact = 7·8 [17]	197·12	$N_x=(1-y/L)N$
4				27·081 Exact = 25·6 [17]	287·02	$N_x=(1-2y/L)N$
5	PQ dof = 46		389·67	±9·3769 Exact = 9·34 [17]	307·74	Constant N_{xy}
6				2·0002 Exact = 2·00 [17]	194·84	Constant N_x, N_y
7				3·4577	209·63	Constant N_x, N_{xy}
8				1·9172	199·33	Constant N_x, N_y, N_{xy}
9	PQ dof = 34		1296·8 (upper–lower bound) = 1294·9 [6–8]	5·3052 Exact = 5·30 [17]	658·94	Constant N_x, N_y
10				4·6929 Exact = 4·5 [17]	704·99	Constant N_x, N_y, N_{xy}
11	Q dof = 14		846·88	14·663 Exact = 12·28 [17]	705·65	Constant N_{xy}
12	PQ dof = 38		839·97	13·370	679·63	Constant N_{xy}
13	Q dof = 14		846·88	15·460	467·54	$N_x=(1-y/L)N$
14				53·145	725·61	$N_x=(1-2y/L)N$
15	Q dof = 18		561·37	12·804	284·03	$N_x=(y/L)N$
16				10·131	282·70	$N_x=(1-y/L)N$
17	Q dof = 25		161·13	6·8378	84·614	$N_x=(y/L)N$

Notation: dof . . . degrees-of-freedom
—— clamped edge ------ simply supported edge —— free edge.

The eigenvalues computed in each case are upper bound, and they are compared with those obtained by other investigators using different methods. The number of degrees-of-freedom, listed in Table 1 refers to the whole plate after imposing the kinematic boundary conditions.

Exact values for examples 13–17 are not available, and the computed results are compared to values obtained graphically from [19]. Examples 2, 5, 11 and 12 indicate the rapid convergence of the element used. It is also clear that the *PQ* arrangement, as suggested in this paper, is superior to the *Q* arrangement presented in [7]. Thus a 4×4 grid for a square clamped plate, using the *Q* arrangement and having 55 degrees-of-freedom, see [7], resulted in the first eigenvalue of $\lambda_v = 1296{\cdot}06$, whereas a 2×2 grid of the same problem involving only 34 degrees-of-freedom in the *PQ* arrangement yielded $\lambda_v = 1296{\cdot}8$ resulting in a substantial saving in computer time, while obtaining essentially the same accuracy.

The arrangements of the elements were so chosen as to preserve the symmetry properties of the plate. It was found that when the arrangement of elements fails to reflect the symmetry of the plate and its boundary conditions, then obviously the eigenmodes do not exhibit the inherent symmetry of the structure either. For example if the direction of the in-plane shear forces acting on a square plate is reversed, then the buckling eigenvalues must be equal and opposite. This should be the case in examples 2 and 5. However, in example 2 if the arrangement of the elements is that shown in Fig. 2c which is not centrally symmetric, the values of the first buckling modes are 9·49 and −10·3. If, however, the centrally symmetrical *Q* or *PQ* arrangements are used, the values of the first buckling made are exactly equal but opposite in sign, as expected considering the complete symmetry of the geometry of the plate and its loading.

Results for both vibration and buckling analysis using the refined 18 degree-of-freedom element are superior to the results of the analyses presented in [10–12, 15, 16].

4. CONCLUSIONS

The buckling stiffness matrix for a refined plate bending element has been derived. This buckling stiffness matrix can take into consideration a variety of in-plane force distributions and supporting conditions. Elements of this buckling matrix are expressed in an explicit form and can be readily used in a finite element computer program for solving plate buckling problems, when the prescribed buckling load is non-uniformly distributed along the edges of the plate. Several examples of buckling, free vibrations, and combined buckling and vibration problems of plates were solved, and the calculated results were compared with the analytical and numerical results obtained using other procedures.

It was found that:

(i) the results obtained using the refined plate bending element is always more accurate than results obtained using larger number of simpler elements when the total number of degrees-of-freedom is the same. Alternatively for a given accuracy the degrees-of-freedom required (or the computational cost) is less if refined elements are used instead of simpler elements;

(ii) if refined elements are used the *Q* arrangement, see Fig. 2, which was shown in [7] to be superior as far as accuracy is concerned is in fact inferior to the *PQ* arrangement suggested in this paper;

(iii) if the plate and its loading is symmetrical and a symmetrical arrangement of elements is used then by reversing the direction of the prescribed shear forces the buckling eigenvalues will be equal and opposite in sign, if the element arrangement is non-symmetrical then the buckling eigenvalues are opposite in sign but will not be equal in magnitude.

REFERENCES

[1] D. von Witt..um, *Berechnung von Platten nach dem Ritzschen Verfahren mit Hilfe dreieckförmiger Maschennetze.* Institut für Statik der Technischen Hochschule Hannover, June (1966).

[2] J. H. Argyris, I. Fried and D. W. Sharpf, The TUBA family of plate elements for the matrix displacement method. *Aeronautical J.*, R.Ae.S. 72, 701–709 (1968).

[3] J. H. Argyris, K. E. Buck *et al.*, Some new elements for the matrix displacement method. *Proc. 2nd Conf. on Matrix Methods in Structural Mechanics.* Wright Patterson Air Force Base, Dayton, Ohio (1968).

[4] K. Bell, *Analysis of Thin Plates in Bending Using Triangular Finite Elements.* Institutt for Statikk, Norges Tekniske Hogskole, Trondheim (1968).

[5] W. Visser, The finite element method in deformation and heat conduction problems. Dr. Ir. Dissertation, T.H. Delft, (1968).

[6] G. R. Cowper, E. Kosko, G. M. Lindenberg and M. D. Olson, Formulation of a new triangular plate bending element. Canadian Aeronautics and Space Institute, *Trans.* 1, 86–90 (1968).

[7] G. R. Cowper, E. Kosko, G. M. Lindenberg and M. D. Olson, A high precision triangular plate bending element. National Research Council of Canada, Aero. Report LR-514 (1968).

[8] G. R. Cowper, E. Kosko, G. M. Lindenberg and M. D. Olson, Static and dynamic application of high precision triangular plate bending element. *AIAA J.* 7, 1957–1965 (1969).

[9] G. A. Butlin and R. Ford, A compatible triangular plate bending finite element. University of Leicester, Engineering Department Report 68-15 (1968).

[10] I. T. Cook and K. C. Rockey, Shear buckling of rectangular plates with mixed boundary conditions. *Aeronautical Quarterly*, 349–356 (1963).

[11] D. J. Dawe, Application of the discrete element method to the buckling anaysis of rectangular plates under arbitrary membrane loading. *Aeronautical Quarterly*, 114–127 (1969).

[12] W. G. Garson and R. E. Newton, Plate buckling analysis using fully compatible finite element. *AIAA J.* 3, 527–529 (1969).

[13] S. M. Dickenson, Lateral vibration of rectangular plates subject to in-plane forces. *J. Sound. Vib.* 16, 465–472 (1971).

[14] S. M. Dickenson and Bassily, Buckling and lateral vibration of rectangular plates subject to in-plane loading—a Ritz approach. *J. Sound Vib.* 24, 219–239 (1972).

[15] R. G. Anderson, B. M. Irons and O. C. Zienkiewicz, Vibration and stability of plates using finite element method. *Int. J. Sol. Struct.* 4, 1031–1055 (1968).

[16] K. K. Kapur and B. J. Hartz, Stability of plates using the finite element method. *J. Engng Mech. Div., ASCE* 92, 177–195 (1966).

[17] S. P. Timoshenko and J. M. Gere, *Theory of Elastic Stability.* Second Edition. McGraw-Hill, New York (1961).

[18] O. C. Zienkiewicz, *The Finite Element Method in Engineering Science.* McGraw-Hill, London (1971).

[19] A. Pflüger, *Stabilitätsprobleme der Elastostatik.* Zweite neubearbeitete Auflage, Springer–Verlag (1964).

(*Received* 26 *February* 1974)

APPENDIX

The buckling stiffness matrix

The coefficients A_1 to C_6 of equation (11) are obtained in terms of the coefficients α_1 to γ_1 of the prescribed in-plane loads by a simple transformation of rotation of axis

$$A_1 = C^2\alpha_1 + 2SC\gamma_1 + S^2\beta_1$$
$$A_2 = C^2\alpha_2 - 2SC\beta_3 + S^2\beta_2$$
$$A_3 = C^2\alpha_3 - 2SC\alpha_2 + S^2\beta_3$$
$$A_4 = C^2\alpha_4 + S^2\beta_4$$
$$A_5 = -SC\beta_4$$

$$
\begin{aligned}
A_6 &= -SC\alpha_4\\
B_1 &= S^2\alpha_1 - 2SC\gamma_1 + C^2\beta_1\\
B_2 &= S^2\alpha_2 + 2SC\beta_3 + C^2\beta_2\\
B_3 &= S^2\alpha_3 + 2SC\alpha_2 + C^2\beta_3\\
B_4 &= S^2\alpha_4 + C^2\beta_4\\
B_5 &= SC\beta_4\\
B_6 &= SC\alpha_4\\
C_1 &= SC(\beta_1 - \alpha_1) + C^2 - S^2)\gamma_1\\
C_2 &= SC(\beta_2 - \alpha_2) - (C^2 - S^2)\beta_3\\
C_3 &= SC(\beta_3 - \alpha_3) - (C^2 - S^2)\alpha_2\\
C_4 &= SC(\beta_4 - \alpha_4)\\
C_5 &= -0{\cdot}5(C^2 - S^2)\beta_4\\
C_6 &= -0{\cdot}5(C^2 - S^2)\alpha_4
\end{aligned}
\tag{16}
$$

where

$$C = \cos\theta \text{ and } S = \sin\theta. \tag{17}$$

Considering the symmetry of the buckling stiffness matrix, i.e. $(K_b)_{ij} = (K_b)_{ji}$, and the loading represented by equation (11), the elements of this matrix are:

$$
\begin{aligned}
(K_b)_{ij} = {} & m_i.m_j[A_1.F(m_{ij}-2, n_{ij}) + A_2\{C.F(m_{ij}-1, n_{ij}) - S.F(m_{ij}-2, n_{ij}+1)\\
& + c_e.F(m_{ij}-2, n_{ij})\} + A_3\{S.F(m_{ij}-1, n_{ij}) + C.F(m_{ij}-2, n_{ij}+1)\\
& + d_e.F(m_{ij}-2, n_{ij})\} + A_4\{S.C.F(m_{ij}, n_{ij}) + (C^2-S^2).F(m_{ij}-1, n_{ij}+1)\\
& - S.C.F(m_{ij}-2, n_{ij}+2) + (S.c_e + C.d_e).F(m_{ij}-1, n_{ij}) + (C.c_e\\
& - S.d_e).F(m_{ij}-2, n_{ij}+1) + c_2.d_e.F(m_{ij}-2, n_{ij})\} + A_5\{C^2.F(m_{ij}, n_{ij})\\
& - 2.S.C.F(m_{ij}-1, n_{ij}+1) + S^2.F(m_{ij}-2, n_{ij}+2) + 2.C.c_e.F(m_{ij}-1, n_{ij})\\
& - 2.S.C.F(m_{ij}-2, n_{ij}+1) + c_e^2.F(m_{ij}-2, n_{ij})\} + A_6\{S^2F(m_{ij}, n_{ij})\\
& + 2.S.C.F(m_{ij}-1, n_{ij}+1) + C^2.F(m_{ij}-2, n_{ij}+2) + 2.S.d_eF(m_{ij}-1, n_{ij})\\
& + 2.c_e.d_e.F(m_{ij}-2, n_{ij}+1) + d_e^2.F(m_{ij}-2, n_{ij})\}] + n_i.n_j[B_1.F(m_{ij}, n_{ij}-2)\\
& + B_2.\{C.F(m_{ij}+1, n_{ij}-2) - S.F(m_{ij}, n_{ij}-1) + c_e.F(m_{ij}, n_{ij}-2)\}\\
& + B_3.\{S.F(m_{ij}+1, n_{ij}-2) + C.F(m_{ij}, n_{ij}-1) + d_e.F(m_{ij}, n_{ij}-2)\}\\
& + B_4.\{S.C.F(m_{ij}+2, n_{ij}-2) + (C^2-S^2).F(m_{ij}+1, n_{ij}-1) - S.C.F(m_{ij}, n_{ij})\\
& + (S.c_e + G.d_e).F(m_{ij}+1, k_{ij}-2) + (C.c_e - S.d_e).F(m_{ij}, n_{ij}-1)\\
& + c_e.d_e.F(m_{ij}, n_{ij}-2)\} + B_5.\{C^2.F(m_{ij}+2, n_{ij}-2) - 2.S.C.F(m_{ij}+1, n_{ij}-1)\\
& + S^2.F(m_{ij}, n_{ij}) + 2.C.c_e.F(M_{ij}+1, N_{ij}-2) - 2.S.c_e.F(m_{ij}, n_{ij}-1)\\
& + c_e2.F(m_{ij}, n_{ij}-2)\} + B_6.\{S^2.F(m_{ij}+2, n_{ij}-2) - 2.S.C.F(m_{ij}+1, n_{ij}-1)\\
& + C.{}^2F(m_{ij}, n_{ij}) + 2.S.d_e.F(m_{ij}+1, n_{ij}-2) + 2.C.d_e.F(m_{ij}, n_{ij}-1)\\
& + d_e^2.F(m_{ij}, n_{ij}-2)\}] + (m_i.n_j + m_j.n_i)[\{C_1.F(m_{ij}-1, n_{ij}+1)\\
& + C_2.\{C.F(m_{ij}, n_{ij}-1) - S.F(m_{ij}-1, n_{ij}) + c_e.F(m_{ij}-1, n_{ij}-1)\}\\
& + C_3.\{S.F(m_{ij}, n_{ij}-1) + C.F(m_{ij}-1, n_{ij}) + d_e.F(m_{ij}-1, n_{ij}-1)\}\\
& + C_4.\{S.C.F(m_{ij}+1, n_{ij}-1) + (C^2-S^2).F(m_{ij}, n_{ij}) - S.C.F(m_{ij}-1, n_{ij}+1)\\
& + (S.c_e + C.d_e).F(m_{ij}, n_{ij}-1) + (C.c_e - S.d_e).F(m_{ij}-1, n_{ij})\\
& + c_e.d_e.F(m_{ij}-1, n_{ij}-1)\} + C_5.\{C^2.F(m_{ij}+1, n_{ij}-1) - 2.S.C.F(m_{ij}, n_{ij})
\end{aligned}
$$

$$+S^2.F(m_{ij}-1,n_{ij}+1)+2.C.c_e.F(m_{ij},n_{ij}-1)-2.S.c_e.F(m_{ij}-1,n_{ij})$$
$$+c_e^2.F(m_{ij}-1,n_{ij}-1)\}+C_6.\{S.^2F(m_{ij}+1,n_{ij}-1)+2.S.C.F(m_{ij},n_{ij})$$
$$+C^2.F(m_{ij}-1,n_{ij}+1)+2.S.d_e.F(m_{ij},n_{ij}-1)+2.c_e.d_e.F(m_{ij}-1,n_{ij})$$
$$+d_e.^2F(m_{ij}-1,n_{ij}-1)\}] \quad (18)$$

where

$$m_{ij}=m_i+m_j \qquad n_{ij}=n_i+n_j$$

and m_i and n_i are the exponents in equation (7).

$$F(m,n)=c^{n+1}[a^{m+1}]-(-b)^{m+1}]m!n!/(m+n+2)!$$

and a, b and c are element dimensions, see Fig. 1. The coefficients A_1 to C_6 and S and C are defined in equations (16) and (17). Finally c_e and d_e denote the coordinates of the origin of the local coordinate system, see Fig. 1.